Optimal reconciliation with immutable forecasts

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Introduction

- Hierarchical time series (HTS) refers to multivariate time series whose observations at every period t respect given linear constraints.
- Forecast reconciliation produces *coherent* forecasts for HTS in a two-step process. First, it generates *incoherent base* forecasts for all time series using arbitrary forecasting models, then reconciles the base forecasts via *projection*.

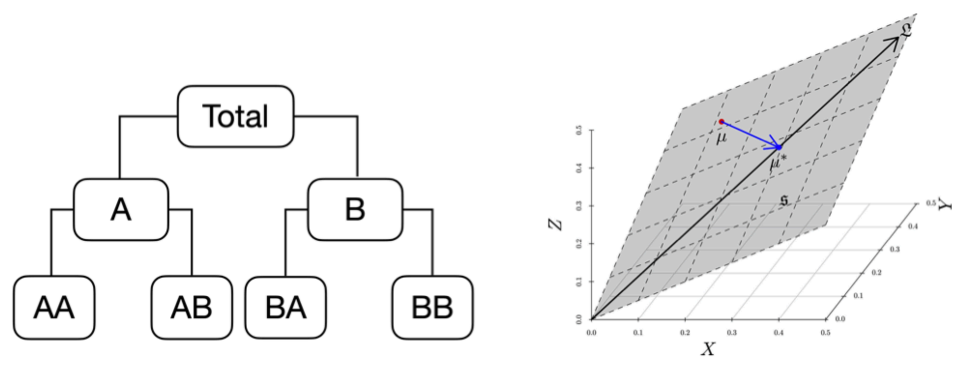

Figure 1: Example of HTS and forecast reconciliation via projection.

- All base forecasts are changed during the reconciliation process, while there may be practical motivations to keep a subset of base forecasts unchanged, or "immutable".
- Examples may include judgmental adjustments by experts, target production of some key product lines and intermittent demand forecasts.

Reconciliation with immutability

We propose a novel method for optimal reconciliation that keeps forecasts of a subset of series immutable. The reconciled forecasts are found as solution to the following optimization problem.

$$\begin{aligned} \min_{\tilde{\boldsymbol{y}}} (\hat{\boldsymbol{y}} - \tilde{\boldsymbol{y}})' \boldsymbol{W}^{-1} (\hat{\boldsymbol{y}} - \tilde{\boldsymbol{y}}) & \\ \text{s.t.} \quad \tilde{\boldsymbol{u}} = \hat{\boldsymbol{u}}, & \\ \boldsymbol{S}_1 \tilde{\boldsymbol{v}} + \boldsymbol{S}_2 \tilde{\boldsymbol{u}} = \tilde{\boldsymbol{w}}, & \end{aligned} \tag{1}$$

where subscripts t are dropped for the ease of exposition.

- $\hat{\boldsymbol{y}} = (\hat{\boldsymbol{w}}', \hat{\boldsymbol{v}}', \hat{\boldsymbol{u}}')'$ is the base forecast.
- $\hat{\boldsymbol{w}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{u}}$ refer to $n - m$ - dimensional non-basis forecasts, $m - k$ - dimensional mutable basis forecasts, and k - dimensional immutable basis forecasts, respectively.
- $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are summing matrices determined by the linear constraints.
- The first constraint ensures immutability, while the second constraint ensures coherency.
- $\boldsymbol{W}$ is the covariance matrix of base forecast error, which can be estimated based on in-sample one-step-ahead forecast error. Estimators could be OLS, structural scaling (WLS$_s$), variance scaling (WLS$_v$) or shrinkage estimator.
- Utilizing $\boldsymbol{W}$ allows us to learn the volatility of the forecasts and dependence structure within the HTS. For example, series with smaller variance will experience smaller change during reconciliation.

This is a generalized least squares problem with a solution

$$\tilde{\boldsymbol{v}} = (\check{\boldsymbol{S}}_1' \boldsymbol{W}_\nu^{-1} \check{\boldsymbol{S}}_1)^{-1} \check{\boldsymbol{S}}_1' \boldsymbol{W}_\nu^{-1} \check{\boldsymbol{\nu}} = \hat{\boldsymbol{G}}_1 \check{\boldsymbol{\nu}}.$$

Then we have unbiased reconciled forecasts

$$\begin{aligned} \tilde{\boldsymbol{b}} = \begin{bmatrix} \tilde{\boldsymbol{v}} \\ \hat{\boldsymbol{u}} \end{bmatrix} = & \begin{bmatrix} \hat{\boldsymbol{G}}_1 & \boldsymbol{0}_{(m-k)\times k} \\ \boldsymbol{0}_{k\times(n-k)} & \boldsymbol{I}_k \end{bmatrix} \\ & \left(\boldsymbol{I}_n - \begin{bmatrix} \boldsymbol{0}_{(n-m)\times(n-k)} & \boldsymbol{S}_2 \\ \boldsymbol{0}_{m\times(n-k)} & \boldsymbol{0}_{m\times k} \end{bmatrix} \right) \begin{bmatrix} \hat{\boldsymbol{w}} \\ \hat{\boldsymbol{v}} \\ \hat{\boldsymbol{u}} \end{bmatrix} = \hat{\boldsymbol{G}} \hat{\boldsymbol{y}} \\ \tilde{\boldsymbol{y}} = \boldsymbol{S} \tilde{\boldsymbol{b}}. & \end{aligned} \tag{2}$$

Immutable Reconciliation with Non-negativity

- Our proposed method for immutable forecasts can be combined with non-negative reconciliation by imposing non-negativity constraints into Equation 1.
- The reconciled forecasts will not still be unbiased when imposing non-negativity constraints.
- Non-negativity constraints can be useful in practical settings such as demand forecasting where observations cannot be negative.

Application: forecasting demand in Tianmao

Dataset

- Product sales data from Tianmao Supermarket, consisting of **40** subcategories (middle level) and **1905** items (bottom level).
- Collected daily from 2019-01-01 to 2021-09-12, but not every time series starts from 2019-01-01 due to the emergence of new products.
- There are spikes on specific dates each year, caused by large-scale promotional events, such as "11.11" or "12.12".
- Our objective is to forecast sales during eight promotional events from the end of 2020 to 2021-09-12 using observations starting from 2019-01-01.

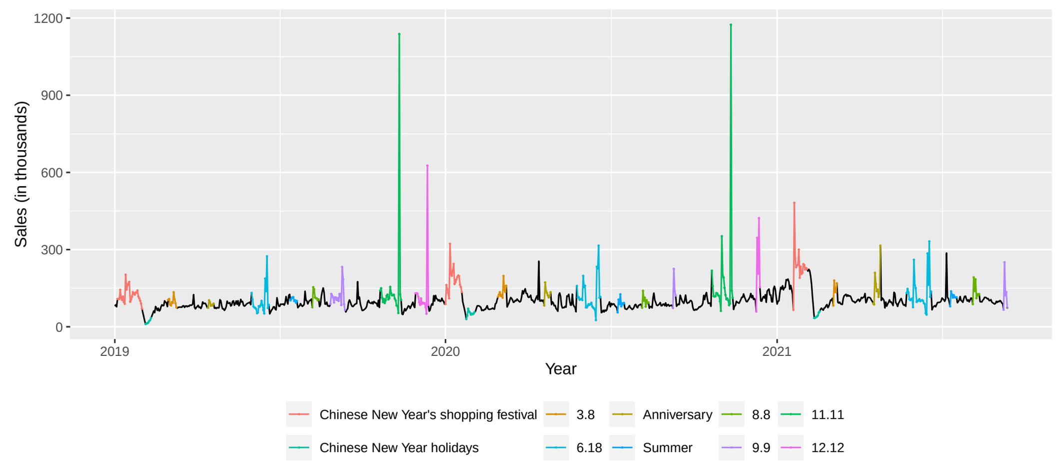

Figure 2: Total sales from 2019-01-01 to 2021-09-12.

Immutable subset selection strategy

- "Top": We keep the base forecasts of top level immutable.
- "Bottom-2": The intermittent series in the bottom level are set to be immutable to reduce the dimension.
- "Bottom-3": We also keep series with more than 365 days of training observations in the bottom level immutable since we are more confident in their base forecasts which are obtained from enough historical information.

Results

Table 1: Out of sample forecasting accuracy.

Level	Base	OLS				WLS$_s$				WLS$_v$			
		C	C+NN	U	U+NN	C	C+NN	U	U+NN	C	C+NN	U	U+NN
Top	2.94	2.94	2.94	2.93	2.92	2.94	2.94	2.72	2.72	2.94	2.94	2.75	2.77
Middle	2.66	9.31	4.94	272.83	48.84	6.41	4.83	16.09	6.50	2.43	2.47	2.39	2.40
Bottom-1	2.04	8.98	4.31	3.98	2.70	7.19	3.71	2.96	2.32	1.97	1.88	1.86	1.83
Bottom-2	0.11	0.11	0.11	42.66	15.43	0.11	0.11	26.99	8.34	0.11	0.11	1.52	1.52
Bottom-3	1.08	1.08	1.08	1.64	1.48	1.08	1.08	1.36	1.25	1.08	1.08	1.58	1.19

- "U" and "C" refer to traditional forecast reconciliation and our proposed method with immutability constraints. "NN" refers to non-negativity constraints.
- Traditional approach ("U" and "U+NN") with OLS and WLS$_s$ produces extremely bad forecasts at the middle and Bottom-2 levels, while our approach mitigates this problem.
- Our approach with WLS$_v$ improves forecast accuracy for mutable series compared to base forecasts, without an associated decrease in accuracy for immutable series.

Conclusion

- We propose a forecast reconciliation approach that can keep the base forecasts of specific levels or multiple nodes from different levels immutable after reconciliation.
- The proposed method is flexible and general enough to allow for expert judgement in choosing the immutable series.
- We prove that the proposed method can produce unbiased reconciled forecasts as long as the base forecasts are unbiased, and the equality constraints do not go beyond the boundary conditions.
- The application to sales data from Tianmao Supermarket shows the potential of the proposed method in reconciling the forecasts of a high-dimensional hierarchy where careful judgement is used in selecting immutable time series.

Papers and code

- Bohan Zhang, Yanfei Kang, Anastasios Panagiotelis and Feng Li (2023).Optimal reconciliation with immutable forecasts. *European Journal of Operational Research*, 308(2), 650-660.
- Code: https://github.com/AngelPone/chf

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